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Long-range one-dimensional Potts models: a cluster mean-field and extrapolation approach

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Abstract. We obtain approximations for the critical temperature (T_c) of q-state ferromagnetic Potts models on one-dimensional lattices with algebraically decaying ferromagnetic pair interactions, i.e. decaying as $1/r^{\theta}$, with $1 < \theta \leqslant 2$. Initially we use a cluster mean-field method to get approximations to the critical temperature and we look at increasingly large clusters. This gives us a sequence of ever more accurate approximate values which can be used as input to various extrapolation algorithms. We see that as the interaction decreases more slowly and the value of q increases we obtain more and more accurate T_c estimates. Our best estimates we believe to be correct to four-figure accuracy.

1. Introduction

Much effort has gone into the investigation of one-dimensional Ising models with algebraically decaying interactions. In a very recent paper Luijten and Blöte [1] in their introductory section gave an excellent overview of the subject and its history over the last 30 years. One of the major advances in the theory of such systems was made by Aizenman *et al* [2]. In their work, which concerned when a phase transition could be proven to be present or not present, they found it useful to broaden the class of models they considered beyond those involving Ising spin variables. Specifically they considered q-state Potts models of which the Ising model is a special case with $q = 2$.

In general, the $q \neq 2$ cases have received much less attention than the $q = 2$ case but very recently there has been an increasing interest in these cases. Most of the work has involved obtaining estimates of the critical temperature and critical exponents. In particular, Glumac and Uzelac [3] have used what they termed a finite-range scaling method, Bernardes and Goulart Rosa Jr [4] have generalized a Bethe approximation approach originally presented for the $q = 2$ case [5] to all q values, and most recently, in this journal, Cannas and de Magalhaes [6] have used a renormalization group approach. In addition to estimates of the critical temperature Glumac and Uzelac [7] have recently shown, using Monte Carlo simulations in combination with finite-size scaling, that for a given q and for infinite-range interactions falling off sufficiently slowly one has a first-order phase transition, while for more rapidly decreasing interaction strengths one has a continuous phase transition.

We will present results which we believe to be more accurate estimates of the critical temperature, for the case of very slowly decaying interactions, than any previously obtained. We will use a cluster mean-field approach in combination with certain extrapolation techniques. We base our belief in the accuracy of our results on two facts. Firstly, there is the very good

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comparison of our results [8] for the $q = 2$ case with those of Luijten and Blöte [1]. Critical temperature estimates in [1] for the special case of the Ising model are accurate to six significant figures. For very slowly decaying interactions our results in [8] match this level of accuracy. Second, Pearce and Griffiths [9] have shown that mean-field approximations become more accurate as q increases and in the limit $q \to \infty$ are exact (see also Katori [10]).

Here we present results for the $q = 2, 3, 4, 5, 8$, and 16 cases. We present the $q = 2$ results only to test the approach by comparing values with those of [1]. In the following section we present the basis of our method along with the necessary notation, while in section 3 we present our results. Some conclusions follow in section 4.

2. Cluster mean-field approximation and extrapolation techniques

We start with a one-dimensional lattice of sites where on the ith site we have a spin variable σ_i where $\sigma_i = 1, 2, \ldots, q$. The Hamiltonian for our q-state Potts model can be written as

$$\mathcal{H} = -\sum_{i<j} \frac{J}{|i-j|^{\theta}} \delta(\sigma_i, \sigma_j) \tag{1}$$

where $|i - j|$ is the distance between sites i and j and with the distance between adjacent sites set equal to one. We consider only the ferromagnetic case, i.e. $J > 0$. The Kronecker delta interaction may be represented in various ways, one of which is by a dot product of two $(q-1)$-dimensional vectors pointing in the q symmetric directions of a $(q-1)$-dimensional hyper-tetrahedron. For more on this formulation of the Hamiltonian and a general review of the Potts model, see Wu [11]. The order parameter for the general Potts model is

$$M_q(\sigma_i) = \frac{q}{q-1} \left\langle \delta(\sigma_i, 1) - \frac{1}{q} \right\rangle \tag{2}$$

where the thermal average of the Kronecker delta function is given in the usual way by

$$\langle \delta(\sigma_i, 1) \rangle = Z^{-1} \sum_{\{\sigma\}} \delta(\sigma_i, 1) \exp(-\beta \mathcal{H}(\{\sigma\})) \tag{3}$$

where Z is the partition function, the sum is over all configurations denoted as $\{\sigma\}$, and $\beta = 1/kT$.

To obtain estimates of the critical temperature for these systems we use a cluster mean-field approach. We treat interactions between two spins in the cluster exactly and we replace all interactions between a site in the cluster, here site i, and a site outside the cluster, here site j, with a mean-field interaction: e.g., $J_{ij}\delta(\sigma_i, \sigma_j)$ is replaced by $J_{ij}\bar{M}_q\delta(\sigma_i, 1)$ where

$$\bar{M}_q = \frac{q}{q-1}\left(\mathcal{M} - \frac{1}{q}\right). \tag{4}$$

Then in the usual mean-field manner we require

$$\langle \delta(\sigma_c, 1) \rangle = \mathcal{M} \tag{5}$$

where c denotes the centre site of the cluster. For the three-site cluster we have

$$\mathcal{H}(\sigma_1, \sigma_2, \sigma_3) = -J[\delta(\sigma_1, \sigma_2) + \delta(\sigma_2, \sigma_3)] - \frac{J}{2^{\theta}}\delta(\sigma_1, \sigma_3) - J\bar{M}_q(\delta(\sigma_1, 1) + \delta(\sigma_3, 1)) \times\left[\sum_{n=1}^{\infty}\frac{1}{n^{\theta}} + \sum_{n=3}^{\infty}\frac{1}{n^{\theta}}\right] - J\bar{M}_q\delta(\sigma_2, 1)\left[2\sum_{n=2}^{\infty}\frac{1}{n^{\theta}}\right]. \tag{6}$$

For an n-site system using the Hamiltonian analogous to that given in (6) one can obtain an expression for the left-hand side of (5) using (3). Then we find all values of $\mathcal{M}$ satisfying (5).

At high temperatures there is only one value of $\mathcal{M}$ satisfying (5) and this gives $\mathcal{M} = 1/q$ which is the disordered state. As we lower the temperature there comes a point when two more solutions of (5) become present. When more than one solution is available one must determine which solution is the 'correct' solution. One way of determining this is to calculate the single-site free energy for each solution and choose the solution which corresponds to the minimum free energy. This is the approach of Kihara *et al* [12]; see also Wu [11]. The calculation of the free energy can be quite difficult. We prefer to think of (5) as a fixed-point equation and at low enough temperatures there will be more than one fixed point. When this is the case one must determine which fixed point is the 'correct' fixed point. Rather than consider the free energy we use a criterion which is much simpler to apply and which was first announced, to our knowledge, in [13]. Specifically, we require the system to go to that solution which has the 'most stable fixed point' of the fixed-point solutions of (5). By 'most stable fixed point' we mean that fixed-point value which has associated with it the minimum absolute value of the derivative with respect to $\mathcal{M}$ of the left-hand side of (5) of all fixed-point solutions to (5). One can easily show that for the one-site cluster, i.e. the standard mean-field approximation first given by Kihara *et al* [12], the results by this method are identical with those given by the free-energy approach. This method has also been shown to be effective when studying the Potts model on the Bethe lattice and gives the same critical temperatures as the free-energy approaches given by di Liberto *et al* [14] and Ananikyan and Akheyan [15]. It has also been shown to deal with Ising spin systems with multi-site interactions [13]. Not having to obtain expressions for free energies and, rather, only having to determine the derivative of the left-hand side of (5) evaluated at the fixed-point value makes the determining of the critical temperatures much easier.

As is typical in the case of mean-field approximations we have, first, that the mean-field critical temperature is too high and, second, that by considering larger and larger clusters we obtain better and better estimates of the critical temperature. Our philosophy in this paper has been to concentrate on the general method and such characteristics as the accuracy of our approximations as a function of q rather than emphasizing a pushing of the method with large computer times. Here we have restricted our calculations to those that can be done on a personal computer. We have used Mathematica for all computations. In the following we use clusters of size 1, 3, 5, 7, and 9 sites for $q = 2$, 3, 4 and 5. As q becomes larger the calculations involved in evaluating an n-site cluster quickly grow. We have looked at $q = 8$ and 16 but have obtained results for 1-, 3-, and 5-site clusters. We have made a much more thorough investigation of the $q = 2$ case [8] using clusters of up to 25 sites and, as stated earlier, present some $q = 2$ results here only because they can be compared with the very accurate results obtained by Luijten and Blöte [1]. Hence they can be used as a general indicator of the accuracy of our methods.

Not only is it true that the larger the cluster size the more accurate the estimate of the true critical temperature one obtains but, furthermore, if estimates are obtained for three different cluster sizes and one assumes a simple power law correction of the form

$$T_c(L) \cong T_c^* + aL^{-\tau} \tag{7}$$

this can be used to get an even better estimate of the critical temperature. In (7) $T_c(L)$ is the critical temperature for the cluster size L, T_c^* is the exact critical temperature and a and τ are constants.

Still greater accuracy can be obtained if more than three $T_c(L)$ are available by applying various extrapolation techniques. In particular, we present results based on the Vanden Broeck and Schwartz transformations (hereafter VBS) [16] introduced into statistical mechanics by Hamer and Barber [17]. The details of this algorithm along with our results are given in the following section.

Table 1. $q = 2$ cluster mean-field critical temperature approximations.

Cluster size (sites)	θ		
	1.1	1.3	1.5
1	$21.168\,8969 < T_c < 21.168\,8970$	$7.863\,8984 < T_c < 7.863\,8985$	$5.224\,7506 < T_c < 5.224\,750\,7$
3	$21.078\,1950 < T_c < 21.078\,1951$	$7.633\,9568 < T_c < 7.633\,9569$	$4.893\,0790 < T_c < 4.893\,079\,1$
5	$21.051\,9341 < T_c < 21.051\,9342$	$7.555\,4363 < T_c < 7.555\,4364$	$4.769\,6027 < T_c < 4.769\,602\,8$
7	$21.039\,3905 < T_c < 21.039\,3906$	$7.514\,3954 < T_c < 7.514\,3955$	$4.701\,7431 < T_c < 4.701\,743\,2$
9	$21.031\,9940 < T_c < 21.031\,9941$	$7.488\,6715 < T_c < 7.488\,6716$	$4.657\,7098 < T_c < 4.665\,777\,10$
	1.7	1.9	2.0
1	$4.108\,5775 < T_c < 4.108\,5776$	$3.499\,4928 < T_c < 3.499\,4929$	$3.289\,8681 < T_c < 3.289\,8682$
3	$3.700\,1256 < T_c < 3.700\,1257$	$3.031\,9264 < T_c < 3.031\,9265$	$2.797\,8432 < T_c < 2.797\,8433$
5	$3.540\,6578 < T_c < 3.540\,6579$	$2.844\,8782 < T_c < 2.844\,8783$	$2.599\,8101 < T_c < 2.599\,8102$
7	$3.450\,3708 < T_c < 3.450\,3709$	$2.737\,1409 < T_c < 2.737\,1410$	$2.485\,1717 < T_c < 2.485\,1718$
9	$3.390\,5201 < T_c < 3.390\,5202$	$2.664\,7585 < T_c < 2.664\,7586$	$2.407\,8178 < T_c < 2.407\,8179$

3. Results

We first present the critical temperature values obtained for clusters of sizes 1, 3, 5, 7, and 9 sites using the mean-field cluster approximation approach. These are given in tables 1–4 for $q = 2$–5, respectively. Similar results are given in tables 5 and 6 for $q = 8$ and $q = 16$ but only for clusters of up to five sites. For the $q = 2$ case we have presented our results using the usual Ising model Hamiltonian

$$\mathcal{H} = -\sum_{i<j} \frac{J}{|i-j|^{\theta}} \sigma_i \sigma_j \tag{8}$$

where $\sigma = \pm 1$. To convert these to critical temperatures which would be obtained using the Pott's model representation for the Hamiltonian, equation (1), one needs to merely divide the critical temperatures given in table 1 by a factor of two. Only results involving up to 5-site clusters rather than up to 25-site clusters given in [8] have been presented because this is the level generally available for the $q > 2$ cases. We have considered six values of θ: 1.1, 1.3, 1.5, 1.7, 1.9, and 2.0. Our results using equation (7) with sequences of site clusters 1, 3, 5 and 3, 5, 7, and 5, 7, 9 are given in tables 7–10 for $q = 2$–5, respectively. These results are labelled FSS results although, technically they do not involve finite-size scaling.

For the case of $q = 2$ and in the region where the system has a classical behaviour, i.e. $1 < \theta \leqslant 1.5$, we can compare our results to those of Luijten and Blöte [1]. One sees that though we have only considered clusters up to nine sites for the very slowly decaying case we obtain reasonable agreement with the results of [1]. For $\theta = 1.1$ we get a four-figure agreement with the results of [1]. In table 7 we have also listed the results from [3, 6]. For all values of θ and using the 5, 7, 9 site cluster sequence our estimates are more accurate than those presented in [6] using the renormalization group. For those of [3], i.e. those found using finite-range scaling, our estimates are more accurate for $\theta = 1.1$, about equally accurate for $\theta = 1.3$, and less accurate for the other θ cases. The decrease in accuracy of our results as θ increases is to be expected as a product of the mean-field approximation but as shown in [9, 10] this will be counteracted to some extent as q increases.

Before going to the $q \neq 2$ cases we wish to present some details of the extrapolation algorithm mentioned at the end of the previous section. The algorithm takes the critical temperature estimates of the various cluster sizes as input and reduces this set of data to a

Table 2. $q = 3$ cluster mean-field critical temperature approximations.

Cluster size (sites)	θ		
	1.1	1.3	1.5
1	$7.635\,0656 < T_c < 7.635\,0657$	$2.836\,3018 < T_c < 2.836\,3019$	$1.884\,4303 < T_c < 1.884\,4305$
3	$7.611\,8335 < T_c < 7.611\,8336$	$2.775\,1018 < T_c < 2.775\,1019$	$1.793\,4827 < T_c < 1.793\,4828$
5	$7.605\,4836 < T_c < 7.605\,4837$	$2.755\,8723 < T_c < 2.755\,8724$	$1.761\,9248 < T_c < 1.761\,9249$
7	$7.602\,5533 < T_c < 7.602\,5534$	$2.746\,3979 < T_c < 2.746\,3980$	$1.745\,4579 < T_c < 1.745\,4580$
9	$7.600\,8675 < T_c < 7.600\,8676$	$2.740\,7293 < T_c < 2.740\,7294$	$1.735\,2139 < T_c < 1.735\,2140$
	1.7	1.9	2.0
1	$1.481\,8561 < T_c < 1.481\,8562$	$1.262\,1752 < T_c < 1.262\,1753$	$1.186\,569\,1 < T_c < 1.186\,659\,2$
3	$1.367\,1248 < T_c < 1.367\,1429$	$1.128\,2163 < T_c < 1.128\,2164$	$1.044\,390\,9 < T_c < 1.044\,391\,0$
5	$1.324\,4864 < T_c < 1.324\,4865$	$1.076\,1482 < T_c < 1.076\,1483$	$0.988\,279\,94 < T_c < 0.988\,279\,95$
7	$1.301\,1781 < T_c < 1.301\,1782$	$1.046\,6966 < T_c < 1.046\,6967$	$0.956\,143\,32 < T_c < 0.956\,143\,33$
9	$1.286\,1575 < T_c < 1.286\,1576$	$1.027\,1798 < T_c < 1.027\,1799$	$0.934\,619\,35 < T_c < 0.934\,619\,36$

Table 3. $q = 4$ cluster mean-field critical temperature approximations.

Cluster size (sites)	θ		
	1.1	1.3	1.5
1	$6.422\,9201 < T_c < 6.422\,9202$	$2.386\,0096 < T_c < 2.386\,0097$	$1.585\,2576 < T_c < 1.585\,2577$
3	$6.407\,1994 < T_c < 6.407\,1995$	$2.343\,6179 < T_c < 2.343\,6180$	$1.521\,0860 < T_c < 1.521\,0861$
5	$6.403\,0214 < T_c < 6.403\,0215$	$2.330\,8998 < T_c < 2.330\,8999$	$1.499\,7424 < T_c < 1.499\,7425$
7	$6.401\,1231 < T_c < 6.401\,1232$	$2.324\,8285 < T_c < 2.324\,8286$	$1.488\,9523 < T_c < 1.488\,9524$
9	$6.400\,0425 < T_c < 6.400\,0426$	$2.321\,2825 < T_c < 2.321\,2826$	$1.482\,4112 < T_c < 1.482\,4113$
	1.7	1.9	2.0
1	$1.246\,5961 < T_c < 1.246\,5962$	$1.061\,791\,8 < T_c < 1.061\,791\,9$	$0.998\,189\,00 < T_c < 0.998\,189\,01$
3	$1.164\,3908 < T_c < 1.164\,3909$	$0.964\,560\,00 < T_c < 0.964\,560\,01$	$0.894\,402\,19 < T_c < 0.894\,402\,20$
5	$1.1347\,868 < T_c < 1.134\,7869$	$0.927\,498\,78 < T_c < 0.927\,498\,79$	$0.854\,010\,89 < T_c < 0.854\,010\,90$
7	$1.118\,9811 < T_c < 1.118\,9812$	$0.906\,819\,96 < T_c < 0.906\,819\,97$	$0.831\,084\,80 < T_c < 0.831\,084\,81$
9	$1.108\,9917 < T_c < 1.108\,9918$	$0.893\,264\,11 < T_c < 0.893\,264\,12$	$0.815\,833\,55 < T_c < 0.815\,833\,56$

single, more accurate, value through a series of steps.

Using the notation of Hamer and Barber [17] one has for the general sequence transformation that, given a sequence of values A_L which converge to a limiting value A_∞, one forms a table of approximants to A_∞ denoted by $[L, N]$ where $[L, 0] = A_L$ and the $(N+1)$th column of approximants is generated from the Nth and $(N-1)$th columns via the formula

$$\frac{1}{[L, N+1] - [L, N]} + \frac{\alpha_N}{[L, N-1] - [L, N]} = \frac{1}{[L+1, N] - [L, N]} + \frac{1}{[L-1, N] - [L, N]} \tag{9}$$

with $[L, -1] \equiv \infty$. Again, following Hamer and Barber, we refer to these approximants as VBS approximants.

The above defines a broad class of transformations based on the definition of α_N. For the case where the sequence converges as

$$A_L \approx A_\infty + b_1 L^{-\lambda_1} + b_2 L^{-\lambda_2} + \cdots \tag{10}$$

Table 4. $q = 5$ cluster mean-field critical temperature approximations.

Cluster size (sites)	θ		
	1.1	1.3	1.5
1	$5.726\,2992 < T_c < 5.726\,2993$	$2.127\,2263 < T_c < 2.127\,2264$	$1.413\,3228 < T_c < 1.413\,3229$
3	$5.714\,2771 < T_c < 5.714\,3772$	$2.094\,5192 < T_c < 2.094\,5193$	$1.363\,1369 < T_c < 1.363\,1370$
5	$5.711\,2665 < T_c < 5.711\,2666$	$2.085\,0183 < T_c < 2.085\,0184$	$1.346\,9549 < T_c < 1.346\,9550$
7	$5.709\,8672 < T_c < 5.709\,8673$	$2.080\,5807 < T_c < 2.080\,5808$	$1.338\,9639 < T_c < 1.338\,9640$
9	$5.709\,0760 < T_c < 5.709\,0761$	$2.078\,0312 < T_c < 2.078\,0313$	$1.334\,2120 < T_c < 1.334\,2121$
	1.7	1.9	2.0
1	$1.111\,3920 < T_c < 1.111\,3921$	$0.946\,6314 < T_c < 0.946\,6315$	$0.889\,9268 < T_c < 0.889\,9269$
3	$1.046\,3696 < T_c < 1.046\,3697$	$0.868\,9742 < T_c < 0.868\,9743$	$0.806\,6764 < T_c < 0.806\,6765$
5	$1.023\,5097 < T_c < 1.023\,5098$	$0.839\,8340 < T_c < 0.839\,8341$	$0.774\,6511 < T_c < 0.774\,6512$
7	$1.011\,5291 < T_c < 1.011\,5292$	$0.823\,7598 < T_c < 0.823\,7599$	$0.756\,6175 < T_c < 0.756\,6176$
9	$1.004\,0738 < T_c < 1.004\,0739$	$0.813\,3193 < T_c < 0.813\,3194$	$0.744\,6945 < T_c < 0.744\,6946$

Table 5. $q = 8$ cluster mean-field critical temperature approximations.

Cluster size (sites)	θ		
	1.1	1.3	1.5
1	$4.662\,2833 < T_c < 4.662\,2834$	$1.731\,9618 < T_c < 1.731\,9619$	$1.150\,7103 < T_c < 1.150\,7104$
3	$4.655\,3130 < T_c < 4.655\,3131$	$1.712\,1454 < T_c < 1.712\,1455$	$1.119\,4543 < T_c < 1.119\,4544$
5	$4.653\,5577 < T_c < 4.653\,5578$	$1.706\,7456 < T_c < 1.706\,7457$	$1.109\,9961 < T_c < 1.109\,9962$
	1.7	1.9	2.0
1	$0.904\,8819 < T_c < 0.904\,8820$	$0.770\,7358 < T_c < 0.770\,7359$	$0.724\,5676 < T_c < 0.724\,5677$
3	$0.863\,4437 < T_c < 0.863\,4438$	$0.720\,2533 < T_c < 0.720\,2534$	$0.669\,9642 < T_c < 0.669\,9643$
5	$0.849\,6063 < T_c < 0.849\,6064$	$0.701\,9715 < T_c < 0.701\,9716$	$0.649\,5276 < T_c < 0.649\,5277$

Table 6. $q = 16$ cluster mean-field critical temperature approximations.

Cluster size (sites)	θ		
	1.1	1.3	1.5
1	$3.647\,9451 < T_c < 3.647\,9452$	$1.355\,1518 < T_c < 1.355\,1519$	$0.900\,3588 < T_c < 0.900\,3589$
3	$3.644\,5841 < T_c < 3.644\,5842$	$1.345\,0668 < T_c < 1.345\,0669$	$0.883\,7778 < T_c < 0.883\,7779$
5	$3.643\,7802 < T_c < 3.643\,7803$	$1.342\,5746 < T_c < 1.342\,5747$	$0.879\,2225 < T_c < 0.879\,2226$
	1.7	1.9	2.0
1	$0.708\,0135 < T_c < 0.708\,0136$	$0.603\,0525 < T_c < 0.603\,0526$	$0.566\,9288 < T_c < 0.566\,9289$
3	$0.685\,2775 < T_c < 0.685\,2776$	$0.574\,5416 < T_c < 0.574\,5417$	$0.535\,6836 < T_c < 0.535\,6837$
5	$0.678\,2743 < T_c < 0.678\,2744$	$0.564\,8170 < T_c < 0.564\,8171$	$0.524\,5462 < T_c < 0.524\,5463$

Barber and Hamer [18] show that a good choice for the value of α_N to be

$$\alpha_N = -\frac{[1-(-1)^N]}{2} \tag{11}$$

for $N = 0, 1, 2, \ldots$.

Table 7. $q = 2$ critical temperature approximations and comparison approximations from [1, 3, 6].

	Finite-size scaling (site clusters)						
θ	1,3,5	3,5,7	5,7,9	VBS algorithm	[1] Monte Carlo simulations	[3] Finite-range scaling	[6] Renormalization group
1.1	20.974 657	20.995 717	20.998 772	20.999 980	21.000 99	21.574	20.80
1.3	7.185 350	7.309 106	7.328 920	7.343 983	7.347 00	7.360	6.96
1.5	3.943 302	4.260 158	4.312 019	4.363 423	4.363 84	4.358	4.00
1.7	2.166 314	2.739 555	2.831 724	2.933 701		2.926	1.56
1.9	0.959 962	1.775 108	1.897 363	2.037 690		2.006	1.54
2.0	0.516 866	1.418 892	1.553 237	1.696 394		1.626	

Table 8. $q = 3$ critical temperature approximations and comparison approximations from [3, 6].

	Finite-size scaling (site clusters)					
θ	1,3,5	3,5,7	5,7,9	VBS algorithm	[3] Finite-range scaling	[6] Renormalization group
1.1	7.5891	7.5935	7.5941	7.594 31	7.353	6.72
1.3	2.6860	2.7095	2.7134	2.716 69	2.589	2.33
1.5	1.6044	1.6637	1.6744	1.685 42	1.663	1.41
1.7	1.041	1.154	1.174	1.1968	1.194	0.95
1.9	0.639	0.816	0.845	0.8785	0.874	0.61
2.0	0.471	0.681	0.713	0.7483	0.742	

Table 9. $q = 4$ critical temperature approximations and comparison approximations from [3, 6].

	Finite-size scaling (site clusters)					
θ	1,3,5	3,5,7	5,7,9	VBS algorithm	[3] Finite-range scaling	[6] Renormalization group
1.1	6.392 84	6.395 55	6.395 88	6.395 99	4.926	5.16
1.3	2.290 06	2.303 73	2.306 03	2.307 84	2.045	1.89
1.5	1.407 68	1.442 36	1.448 75	1.455 21	1.403	1.14
1.7	0.9639	1.0312	1.0437	1.0577	1.048	0.78
1.9	0.6484	0.7592	0.7780	0.7996	0.797	0.51
2.0	0.5124	0.6474	0.6689	0.6918	0.694	

Using the VBS algorithm, we see from table 7 that, now, from our five-cluster sequence we obtain an estimate of T_c which agrees with the results of [1] to five-figure accuracy for $\theta = 1.1$ rather than the four-figure agreement found using the finite-scaling-like approach. Going on to $\theta = 1.3$ and 1.5 we have at least three-figure agreement which betters the results of [3] for these two θ values. Since Luijten and Blöte only consider the 'classical regime' of the one-dimensional ferromagnetic Ising model, that is those θ values where the critical exponents take on their classical values, we can make no comparison beyond $\theta = 1.5$. We have no doubt that our results become less accurate as θ increases. It should be pointed out that the difference between the results using equation (7) and the VBS algorithm increase as θ increases.

The particular case of $\theta = 2.0$ has received by far the most attention, and while estimates having the five- and six-figure accuracy of the 'classical regime' are not available, to gain some idea of the accuracy of our method in this case we quote the following results. Monte Carlo

Table 10. $q = 5$ critical temperature approximations and comparison approximations from [3] and the mean-field approximation.

	Finite-size scaling (site clusters)				[3]	
θ	1,3,5	3,5,7	5,7,9	VBS algorithm	Finite-range scaling	Mean-field results
1.1	5.703 96	5.705 85	5.706 11	5.706 18	3.57	5.726
1.3	2.056 84	2.066 26	2.067 83	2.069 00	1.736	2.127
1.5	1.283 58	1.307 54	1.312 00	1.316 38	1.245	1.413
1.7	0.903 64	0.950 83	0.959 54	0.969 63	0.956	1.111
1.9	0.637 22	0.716 83	0.730 76	0.746 73	0.745	0.9466
2.0	0.521 15	0.619 85	0.635 89	0.653 02	0.659	0.8899

Table 11. $q = 8$ and 16 critical temperature approximations and comparison approximations from [3] and the mean-field approximation.

θ	1,3 and 5 site clusters	VBS algorithm	[3] Finite-range scaling	Mean-field results
(*a*)				
1.1	4.6497	4.6530	2.08	4.6623
1.3	1.6929	1.7047	1.32	1.7320
1.5	1.0789	1.1059	1.01	1.1507
1.7	0.7891	0.8427	0.806	0.9049
1.9	0.5939	0.6915	0.649	0.7707
2.0	0.5092	0.6373	0.589	0.7246
(*b*)				
1.1	3.6422	3.6435	1.28	3.6479
1.3	1.3373	1.3418	0.98	1.3552
1.5	0.8673	0.8775	0.79	0.9004
1.7	0.6542	0.6752	0.649	0.7080
1.9	0.5192	0.5598	0.543	0.6031
2.0	0.4629	0.5184	0.5008	0.5669

results of Bhattacharjee *et al* [19] give $T_c \approx 1.587$ and high-temperature series expansion results of Matvienko [20] give $T_c \approx 1.522$. Many other estimates can be found in [21] where the present author has obtained a rigorous upper bound of the critical temperature which shows $T_c < 1.636$. The procedures used in [6] give $T_c = \pi^2/6 \approx 1.645$ for the $\theta = 2.0$ case and [3] has $T_c \approx 1.626$, while our results are $T_c \approx 1.552$ using the best of the results from equation (7) and $T_c \approx 1.696$ using the VBS algorithm. The fact that our two results, one based on the 5, 7, 9 sequence with equation (5) and the other on the VBS algorithm, differ by such a large amount indicates the lack of accuracy by the time we approach $\theta = 2.0$.

Based on the $q = 2$ results in comparison with those of [1] and with the results of [9, 10] indicating that we should have greater accuracy as q increases, we believe, for $\theta = 1.1$ we have at least four-figure accuracy when $q > 2$ and that the accuracy decreases to three figures by the time θ has increased to 1.5. When we go to the cases of $q = 8$ and $q = 16$ (see table 11) we are hampered by the memory requirements of our computer methods and have only been able to consider up to five-site clusters. Nevertheless, we believe our results for the $\theta < 1.5$ cases to be more accurate than any previously obtained.

4. Conclusions

With any approximation, the accuracy of the results and the amount of work necessary to obtain them are the two important characteristics. We have shown that a cluster mean-field method in conjunction with various extrapolation methods is likely to provide the most accurate estimates currently available of the critical temperature of Potts models on one-dimensional lattices and having ferromagnetic algebraically decaying interactions with $\theta \leqslant 1.5$. Also, our results seem to show that the finite-range method works best when the above method works poorest. The fact that all the work reported was done on a personal computer and done in a matter of minutes or hours indicates that this method does not require a significant amount of hardware or undue amount of time.

Equally importantly, we see that the method of determining the transition temperatures based on the stability of the fixed points which was motivated by a dynamical-systems study of the Potts model on the Bethe lattice [13] is clearly of use here. As stated earlier for the one-site cluster, i.e. the usual mean-field approximation, we know by direct comparison with the results of Kihara *et al* [12] for the next-nearest-neighbour case in any dimension that this method exactly duplicates the usual free-energy approach. This criteria has allowed us to by-pass the rather complicated computation of a free energy. This then gives us a class of systems, in addition to those listed in [13] where the approach is successful.

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